

Cosmological perturbations of self-accelerating universe in nonlinear massive gravity

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Abstract

We study cosmological perturbations of self-accelerating universe solutions in the recently proposed nonlinear theory of massive gravity, with general matter content. While the broken diffeomorphism invariance implies that there generically are 2 tensor, 2 vector and 2 scalar degrees of freedom in the gravity sector, we find that the scalar and vector degrees have vanishing kinetic terms and nonzero mass terms. Depending on their nonlinear behavior, this indicates either non-dynamical nature of these degrees or strong couplings. Assuming the former, we integrate out the 2 vector and 2 scalar degrees of freedom. We then find that in the scalar and vector sectors, gauge-invariant variables constructed from metric and matter perturbations have exactly the same quadratic action as in general relativity. The difference from general relativity arises only in the tensor sector, where the graviton mass modifies the dispersion relation of gravitational waves, with a time-dependent effective mass. This may lead to modification of stochastic gravitational wave spectrum.

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I. INTRODUCTION

The stability of general relativity (GR) predictions against small graviton mass has been a persistent challenge of classical field theory. The simplest ghost-free extension of GR with a linear mass term [1] suffers from the van Dam-Veltman-Zakharov discontinuity, giving rise to different predictions for the classical tests in the vanishing mass limit [2, 3]. Although this problem can be alleviated by nonlinear terms [4], the cost is the emergence of the Boulware-Deser (BD) ghost [5], which is generically unavoidable due to six degrees of freedom in the metric instead of the five of the massive spin 2 field.

Adopting an effective field theory approach in the decoupling limit, the source of all the above issues can be traced back to the helicity 0 mode of the graviton [6]. In this perspective, an analogue of the cancellation of the BD ghost in the linear theory by a specific choice of the mass term can be performed, rendering the nonlinear theory ghost-free up to quartic order by tuning the coefficients [7].

More recently, a two parameter theory of nonlinear massive gravity was developed [8, 9]. In this construction, terms to each order are chosen to remove the additional degree of freedom (would-be BD ghost) at the decoupling limit and it has the potential to be free of the BD ghost at fully nonlinear level [10–12]. The more general construction with dynamical auxiliary metrics was also claimed to be free from the BD ghost at fully nonlinear order [13, 14].

The massive extensions of GR are known to allow self-accelerating solutions [15–29]. This provides an application opportunity for the recent formulations of massive gravity, as an alternative approach to account for the current accelerated expansion. On the other hand, to model the accelerating universe in the scope of this infrared modified gravity theory, a cosmological solution is necessary. In the construction of [8, 9] with Minkowski fiducial metric, flat Friedmann-Robertson-Walker (FRW) universe cannot be realized [22], although cosmological solutions with negative spatial curvature exist [23]. For more general constructions with a nondynamical [30] and dynamical [27, 28] auxiliary metrics, maximally symmetric FRW with either curvature may be allowed.

Given the nontrivial cosmological solutions which self-accelerate, it is thus a necessity to understand their properties against perturbations. In the light of the construction which removes the additional degree (would-be BD ghost), the expectation is to have five degrees

of freedom associated with the five polarizations of the massive graviton. On the other hand, the absence of BD ghost does not guarantee that the theory is safe; one still needs to determine the conditions under which the helicity 0 and 1 modes, which are pure gauge in the massless theory, are stable [31]. Furthermore, these additional degrees should not be in conflict with observations. For instance, because of the emergence of a new scalar degree in the gravity sector, the Newtonian potential may acquire modifications from couplings between the matter sector and the helicity 0 graviton. If this is the case, the parameters of the theory can be restricted by e.g. the solar system tests.

The primary goal of the present paper is to address these questions. We consider the cosmological solutions in massive gravity as background and in the presence of a generic matter content, we present a gauge invariant formulation of perturbations. Although as we argued above, we expect the gravity sector to contain 5 dynamical degrees of freedom, at the level of the quadratic action, we show that the helicity 0 and 1 graviton modes have vanishing kinetic terms but finite masses.¹ As a result, in addition to the matter perturbations, the only dynamical degrees of freedom are the two tensor polarizations of gravity waves.

This is exactly the same number of degrees of freedom as in GR. For the action quadratic in perturbations around the cosmological backgrounds, the mass terms of the potentially ghost-free construction turn out to be completely decoupled from the standard part (Einstein-Hilbert and matter terms) for scalar and vector modes. Thus, these modes evolve identically to their counterparts in GR, except for the additional cosmological constant contributed by the graviton mass. The only non trivial effect on the dynamics occurs in the tensor modes, which acquires a time dependent mass term determined by the fiducial metric of the theory.

The paper is organized as follows. In Section II, we review the setup with a fiducial metric that is Minkowski and summarize the only cosmological solution [23] allowed in this case. In Section III, we give a detailed study of the complete quadratic action for perturbations in the presence of a general fiducial metric of the FRW form, with an ansatz for the physical background metric also of FRW type with arbitrary spatial curvature. As an example, we specify to perturbations around the open universe solution [23] driven by a single scalar field matter in Section IV. We conclude with Section V, where we summarize our results.

¹ A similar situation was noted in the decoupling limit [17, 24].

The paper is supplemented by a number of Appendices, where the details of calculations are presented.

II. OPEN FRW SOLUTION WITH MINKOWSKI FIDUCIAL METRIC

In this section, we review the open FRW universe solution [23] in nonlinear massive gravity [9] coupled to general matter content.

The covariant action for the gravity sector is constructed out of the four dimensional metric $g_{\mu\nu}$ and the four scalar fields ϕ^a ($a = 0, 1, 2, 3$) called *Stückelberg fields*. The action respects the Poincare symmetry in the field space, i.e. invariance under the constant shift of each of ϕ^a and the Lorentz transformation mixing them:

$$\phi^a \rightarrow \phi^a + c^a, \quad \phi^a \rightarrow \Lambda_b^a \phi^b. \quad (1)$$

The following line element in the field space is invariant under these transformations.

$$\eta_{ab} d\phi^a d\phi^b = -(d\phi^0)^2 + \delta_{ij} d\phi^i d\phi^j. \quad (2)$$

Indeed, this is the unique geometrical quantity in the field space of ϕ^a . Thus the action can depend on ϕ^a only through the spacetime tensor

$$f_{\mu\nu} \equiv \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b. \quad (3)$$

In this language, general covariance is spontaneously broken by the vacuum expectation value (vev) of $f_{\mu\nu}$. By assumption, matter fields propagate on the physical metric $g_{\mu\nu}$, but are not coupled to $f_{\mu\nu}$ directly. The tensor $f_{\mu\nu}$, constructed from the invariant line element in the field space, is often called a *fiducial metric*. On the other hand, the spacetime metric $g_{\mu\nu}$, on which matter fields propagate, is often called a *physical metric*.

The gravity action is the sum of the Einstein-Hilbert action (with the cosmological constant Λ) $I_{EH,\Lambda}$ for the physical metric $g_{\mu\nu}$ and the graviton mass term I_{mass} specified below. Adding the matter action I_{matter} , the total action is

$$I = I_{EH,\Lambda}[g_{\mu\nu}] + I_{mass}[g_{\mu\nu}, f_{\mu\nu}] + I_{matter}[g_{\mu\nu}, \sigma_I], \quad (4)$$

where

$$I_{EH,\Lambda}[g_{\mu\nu}] = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad (5)$$

$$I_{mass}[g_{\mu\nu}, f_{\mu\nu}] = M_{Pl}^2 m_g^2 \int d^4x \sqrt{-g} (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4), \quad (6)$$

and $\{\sigma_I\}$ ($I = 1, 2, \dots$) represent matter fields. Demanding the absence of ghost at least in the decoupling limit [9], each contribution in the mass term I_{mass} is constructed as

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2} ([\mathcal{K}]^2 - [\mathcal{K}^2]) , \\ \mathcal{L}_3 &= \frac{1}{6} ([\mathcal{K}]^3 - 3 [\mathcal{K}] [\mathcal{K}^2] + 2 [\mathcal{K}^3]) , \\ \mathcal{L}_4 &= \frac{1}{24} ([\mathcal{K}]^4 - 6 [\mathcal{K}]^2 [\mathcal{K}^2] + 3 [\mathcal{K}^2]^2 + 8 [\mathcal{K}] [\mathcal{K}^3] - 6 [\mathcal{K}^4]) ,\end{aligned}\tag{7}$$

where the square brackets denote trace operation and

$$\mathcal{K}_\nu^\mu = \delta_\nu^\mu - \left(\sqrt{g^{-1}f} \right)_\nu^\mu .\tag{8}$$

The square-root in this expression is the positive definite matrix defined through

$$\left(\sqrt{g^{-1}f} \right)_\rho^\mu \left(\sqrt{g^{-1}f} \right)_\nu^\rho = f_\nu^\mu \ (\equiv g^{\mu\rho} f_{\rho\nu}).\tag{9}$$

As already stated above, a vev of the tensor $f_{\mu\nu}$ breaks general covariance spontaneously. Thus, in order to find FRW cosmological solutions in this theory, we should adopt an ansatz in which not only $g_{\mu\nu}$ but also $f_{\mu\nu}$ respects the symmetry of the FRW universes [23]. Since the tensor $f_{\mu\nu}$ is the pullback of the Minkowski metric in the field space to the physical spacetime, construction of such an ansatz is equivalent to finding a flat, closed, or open FRW coordinate system for the Minkowski line element. It is well known that the Minkowski line element does not admit a closed FRW chart but allows an open FRW chart. For this reason, in order to find open FRW solutions [23], we first perform the field redefinition from ϕ^a to new fields φ^a so that $f_{\mu\nu}$ written in terms of φ^a manifestly has the symmetry of open FRW universes as

$$f_{\mu\nu} = -n^2(\varphi^0) \partial_\mu \varphi^0 \partial_\nu \varphi^0 + \alpha^2(\varphi^0) \Omega_{ij}(\varphi^k) \partial_\mu \varphi^i \partial_\nu \varphi^j ,\tag{10}$$

where $i, j = 1, 2, 3$, and

$$\Omega_{ij}(\varphi^k) = \delta_{ij} + \frac{K \delta_{il} \delta_{jm} \varphi^l \varphi^m}{1 - K \delta_{lm} \varphi^l \varphi^m}\tag{11}$$

is the metric of the maximally symmetric space with the curvature constant K (< 0).

Concretely, this is achieved by

$$\phi^0 = f(\varphi^0) \sqrt{1 - K \delta_{ij} \varphi^i \varphi^j} , \quad \phi^i = \sqrt{-K} f(\varphi^0) \varphi^i ,\tag{12}$$

and

$$n(\varphi^0) = |\dot{f}(\varphi^0)| , \quad \alpha(\varphi^0) = \sqrt{-K} |f(\varphi^0)| ,\tag{13}$$

where f is a function to be determined and \dot{f} represents its derivative. We then adopt the “unitary gauge”

$$\varphi^0 = t, \quad \varphi^i = x^i, \quad (14)$$

so that

$$f_{\mu\nu} dx^\mu dx^\nu = -(\dot{f}(t))^2 dt^2 + |K| (f(t))^2 \Omega_{ij}(x^k) dx^i dx^j. \quad (15)$$

This is nothing but the Minkowski line element in the open chart. For the physical metric, we adopt the open FRW ansatz

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij}(x^k) dx^i dx^j. \quad (16)$$

Hereafter, we assume that $N > 0$ and $a > 0$, without loss of generality.

The background action now yields, up to boundary terms,

$$I = M_{Pl}^2 \int dt d^3x N a^3 \sqrt{\Omega} (L_{EH}[N, a] + m_g^2 L_{mass}[N, a, f]) + I_{matter}[N, a, \sigma_I], \quad (17)$$

consisting of the Einstein-Hilbert part

$$L_{EH} = \frac{3K}{a^2} - \frac{3\dot{a}^2}{a^2 N^2}, \quad (18)$$

and the contribution from the mass term

$$\begin{aligned} L_{mass} = & \left(1 - \frac{\sqrt{-K}|f|}{a}\right) \left[6 + 4\alpha_3 + \alpha_4 - \frac{\sqrt{-K}|f|}{a} (3 + 5\alpha_3 + 2\alpha_4) - \frac{K|f|^2}{a^2} (\alpha_3 + \alpha_4)\right] \\ & + \text{sgn}(\dot{f}/f) \frac{|f|\dot{a}}{N a} \\ & \times \left[3(3 + 3\alpha_3 + \alpha_4) - \frac{3\sqrt{-K}|f|}{a} (1 + 2\alpha_3 + \alpha_4) - \frac{K|f|^2}{a^2} (\alpha_3 + \alpha_4)\right]. \end{aligned} \quad (19)$$

Hereafter, an overdot represents derivative w.r.t. the time t .

Varying the action (17) with respect to f yields the following constraint

$$\begin{aligned} & \left[H - \text{sgn}(\dot{f}/f) \frac{\sqrt{-K}}{a}\right] \\ & \times \left[3 + 3\alpha_3 + \alpha_4 - \frac{2\sqrt{-K}|f|}{a} (1 + 2\alpha_3 + \alpha_4) - \frac{K|f|^2}{a^2} (\alpha_3 + \alpha_4)\right] = 0, \end{aligned} \quad (20)$$

where the Hubble expansion rate of the physical metric is defined as

$$H \equiv \frac{\dot{a}}{N a}. \quad (21)$$

Out of the three solutions of the constraint (20), the trivial solution $\dot{a} = \text{sgn}(\dot{f}/f)\sqrt{-K} N$ corresponds to the Minkowski spacetime in open chart. The remaining two branches of solutions are given by [23]

$$\alpha(t) = X_{\pm} a(t), \quad X_{\pm} \equiv \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4} (> 0), \quad (22)$$

and describe FRW cosmologies with $K < 0$.² In the present paper we will focus only on these nontrivial cosmological solutions.

Using the above constraint and varying the action (17) with respect to N and a , we obtain the remaining background equations

$$\begin{aligned} 3H^2 + \frac{3K}{a^2} &= \Lambda_{\pm} + \frac{1}{M_{Pl}^2} \rho, \\ -\frac{2\dot{H}}{N} + \frac{2K}{a^2} &= \frac{1}{M_{Pl}^2} (\rho + P), \end{aligned} \quad (23)$$

where ρ and P are the energy density and the pressure of matter fields calculated from I_{matter} , and

$$\Lambda_{\pm} \equiv -\frac{m_g^2}{(\alpha_3 + \alpha_4)^2} \left[(1 + \alpha_3) (2 + \alpha_3 + 2\alpha_3^2 - 3\alpha_4) \pm 2 (1 + \alpha_3 + \alpha_3^2 - \alpha_4)^{3/2} \right]. \quad (24)$$

Thus, for the cosmological solutions (22), the contribution from the graviton mass term I_{mass} at the background level mimics a cosmological constant with the value Λ_{\pm} .

For $\alpha_4 = (3 + 2\alpha_3 + 3\alpha_3^2)/4$ and $\pm(1 + \alpha_3) > 0$, the effective cosmological constant Λ_{\pm} vanishes, and the background solution reduces to the open FRW universe solution of GR. On the other hand, both X_{\pm} and Λ_{\pm} diverge for $\alpha_4 = -\alpha_3$ and $\pm(1 + \alpha_3) > 0$. In Figure 1, we show the sign of Λ_{\pm} in the (α_3, α_4) space. Note that X_{\pm} are restricted to be positive by definition, as explained in footnote 2. Except for the restriction due to the positivity of X_{\pm} , these are in agreement with the analogous region plots presented in Ref.[24]³.

² Note that X_{\pm} are positive by definition since $\alpha(t) > 0$ and we assumed $a(t) > 0$. If we instead assumed $a(t) < 0$ then the corresponding solutions would be $\alpha(t) = -X_{\pm} a(t)$ with the same X_{\pm} and we would conclude $X_{\pm} > 0$ again. The essential reason for the positivity of X_{\pm} is that the square-root in (8) is the positive one.

³ Substituting $\alpha_3 \rightarrow 3\alpha_3$, $\alpha_4 \rightarrow 12\alpha_4$, and switching the positive and negative branch definitions, our expression (24) recovers Eq.(6.6) of Ref.[24]. However, note that $f_{\mu\nu}$ in the solution of [24] does not respect the FRW symmetry.

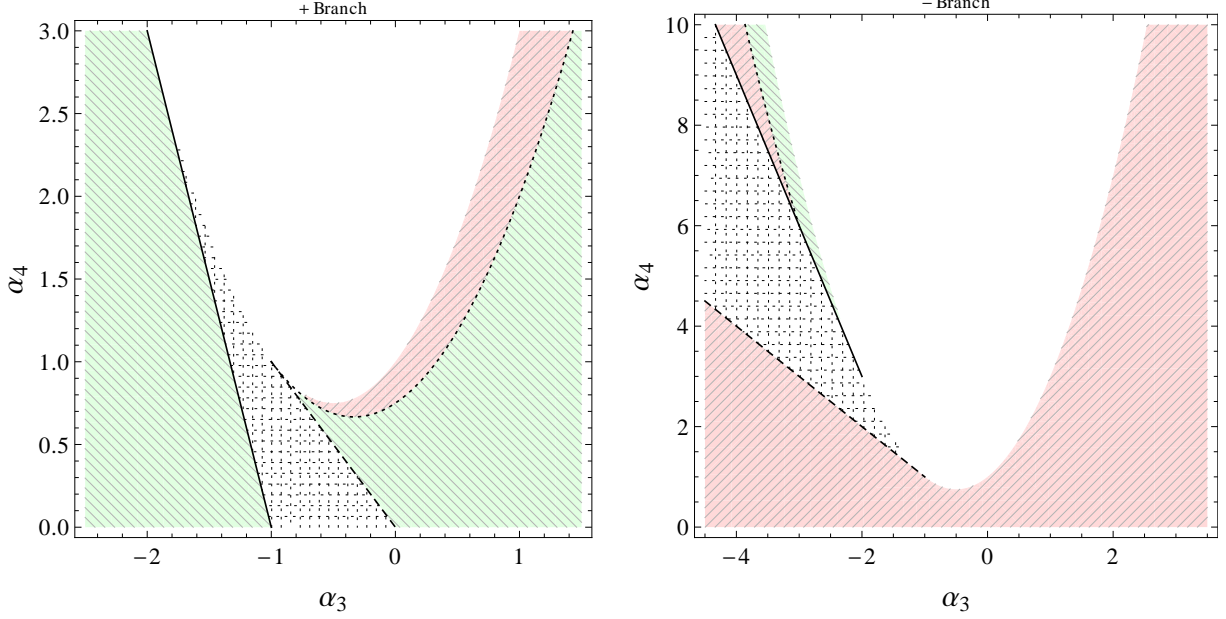


FIG. 1: Sign of the effective cosmological constant Λ_{\pm} in the positive (left panel) and negative (right panel) branches. In the red (green) region with $+45^{\circ}$ (-45°) lines, Λ_{\pm} is positive (negative). The white region and the dotted squared region correspond to $1 + \alpha_3 + \alpha_3^2 - \alpha_4 < 0$ and $X_{\pm} < 0$, respectively, and are excluded since the cosmological solutions (22) do not exist there. Along the dotted black line (defining the boundary between the red and green regions), $\Lambda_{\pm} = 0$ and the background solution reduces to the GR one. The solid line corresponds to $X_{\pm} = 0$ and thus defines one of the boundaries between the allowed (red or green) and excluded (dotted squared) regions. Along the dashed line, both X_{\pm} and Λ_{\pm} diverge, and it defines another boundary between the allowed (red or green) and excluded (dotted squared) regions.

III. PERTURBATIONS IN GENERAL SETUP

In this section we consider the graviton mass term $I_{mass}[g_{\mu\nu}, f_{\mu\nu}]$ defined by (6)-(8) and (10)-(11), but with an arbitrary value of K and arbitrary functions $n(\varphi^0)$ and $\alpha(\varphi^0)$. We shall develop a formalism to analyze perturbations of this generalized system around flat ($K = 0$), closed ($K > 0$) and open ($K < 0$) FRW universes. Cosmological implications of this type of generalized massive gravity will be discussed in future publication.

For the background we adopt the physical metric of the FRW form (16) with (14), but with general K , $n(\varphi^0)$ and $\alpha(\varphi^0)$. Without loss of generality, we assume that $N > 0$, $n > 0$,

$a > 0$ and $\alpha > 0$ at least in the vicinity of the time of interest, where N and a are the lapse function and the scale factor of the background FRW physical metric. (Otherwise, we consider $|N|$, $|n|$, $|a|$ and $|\alpha|$ and rename them as N , n , a and α .) As reviewed in the previous section for open universes in the case of the Minkowski fiducial metric and as shown in Appendix A for general cases with arbitrary K , $n(\varphi^0)$ and $\alpha(\varphi^0)$, the background equation of motion for the Stückelberg fields φ^a has three branches of solutions. One of them does not allow nontrivial cosmologies and thus is not of our interest. The other two branches of solutions allow nontrivial cosmologies and are given by (22) even for general K , $n(\varphi^0)$ and $\alpha(\varphi^0)$. In this section we then consider perturbations of the physical metric and the Stückelberg fields around the FRW solutions in these nontrivial branches.

A. Exponential map and Lie derivative

Since the fiducial metric $f_{\mu\nu}$ is defined as in (10) without referring to the physical metric $g_{\mu\nu}$, let us begin with perturbations of the Stückelberg fields φ^a . We define perturbations π^a of φ^a through the so-called exponential map. Actually, since the action will be expanded only up to the quadratic order, we can truncate the exponential map at the second order. We thus define π^a by

$$\varphi^a = x^a + \pi^a + \frac{1}{2}\pi^b\partial_b\pi^a + O(\epsilon^3), \quad (25)$$

or equivalently,

$$\pi^a = (\varphi^a - x^a) - \frac{1}{2}(\varphi^b - x^b)\partial_b(\varphi^a - x^a) + O(\epsilon^3). \quad (26)$$

Here, ϵ is a small number counting the order of perturbative expansion: $\pi^a = O(\epsilon)$ and $\varphi^a - x^a = O(\epsilon)$. By substituting the expansion (25) to the definition of the fiducial metric

$$f_{\mu\nu} = \bar{f}_{ab}(\varphi^c)\partial_\mu\varphi^a\partial_\nu\varphi^b, \quad (27)$$

where

$$\bar{f}_{00}(\varphi^c) = -n^2(\varphi^0), \quad \bar{f}_{0i}(\varphi^c) = \bar{f}_{i0}(\varphi^c) = 0, \quad \bar{f}_{ij}(\varphi^c) = \alpha^2(\varphi^0)\Omega_{ij}(\varphi^k), \quad (28)$$

we obtain

$$f_{\mu\nu} = \bar{f}_{\mu\nu}(x^\rho) + \mathcal{L}_\pi\bar{f}_{\mu\nu}(x^\rho) + \frac{1}{2}(\mathcal{L}_\pi)^2\bar{f}_{\mu\nu}(x^\rho) + O(\epsilon^3). \quad (29)$$

Here, \mathcal{L}_π represents the Lie derivative along π^μ . Actually, this formula is not restricted to (28) but holds for any $\bar{f}_{ab}(\varphi^c)$.

B. Stückelberg fields and gauge invariant variables

We define perturbations ϕ , β_i and h_{ij} of the physical metric by

$$\begin{aligned} g_{00} &= -N^2(t) [1 + 2\phi], \\ g_{0i} &= N(t)a(t)\beta_i \\ g_{ij} &= a^2(t) [\Omega_{ij}(x^k) + h_{ij}]. \end{aligned} \quad (30)$$

We suppose that $\phi, \beta_i, h_{ij} = O(\epsilon)$.

Under the linear gauge transformation

$$x^\mu \rightarrow x^\mu + \xi^\mu, \quad (\xi^\mu = O(\epsilon)) \quad (31)$$

each variable transforms as

$$\begin{aligned} \pi^0 &\rightarrow \pi + \xi^0, \\ \pi_i &\rightarrow \pi_i + \xi_i, \\ \phi &\rightarrow \phi + \frac{1}{N}\partial_t(N\xi^0), \\ \beta_i &\rightarrow \beta_i - \frac{N}{a}D_i\xi^0 + \frac{a}{N}\dot{\xi}_i, \\ h_{ij} &\rightarrow h_{ij} + D_i\xi_j + D_j\xi_i + 2NH\xi^0\Omega_{ij}, \end{aligned} \quad (32)$$

where H is the Hubble expansion rate as defined in (21),

$$\pi_i \equiv \Omega_{ij}\pi^j, \quad \xi_i \equiv \Omega_{ij}\xi^j, \quad (33)$$

and D_i is the spatial covariant derivative compatible with Ω_{ij} .

We then define gauge invariant variables

$$\begin{aligned} \phi^\pi &\equiv \phi - \frac{1}{N}\partial_t(N\pi^0), \\ \beta_i^\pi &\equiv \beta_i + \frac{N}{a}D_i\pi^0 - \frac{a}{N}\dot{\pi}_i, \\ h_{ij}^\pi &\equiv h_{ij} - D_i\pi_j - D_j\pi_i - 2NH\pi^0\Omega_{ij}. \end{aligned} \quad (34)$$

For later convenience, let us decompose β_i^π and h_{ij}^π as

$$\begin{aligned} \beta_i^\pi &= D_i\beta^\pi + S_i^\pi, \\ h_{ij}^\pi &= 2\psi^\pi\Omega_{ij} + \left(D_iD_j - \frac{1}{3}\Omega_{ij}\Delta\right)E^\pi + \frac{1}{2}(D_iF_j^\pi + D_jF_i^\pi) + \gamma_{ij}, \end{aligned} \quad (35)$$

where S_i^π and F_i^π are transverse, and γ_{ij} is transverse and traceless:

$$D^i S_i^\pi = D^i F_i^\pi = 0, \quad D^i \gamma_{ij} = 0, \quad \Omega^{ij} \gamma_{ij} = 0, \quad (36)$$

and $D^i \equiv \Omega^{ij} D_j$.

C. Graviton mass term

At the FRW background level, the graviton mass term acts as an effective cosmological constant Λ_\pm shown in (24). The proof of this statement is presented in Appendix A for arbitrary K , $n(\varphi^0)$ and $\alpha(\varphi^0)$. Thus, calculations are expected to be simplified if we add $M_{Pl}^2 \int d^4x \sqrt{-g} \Lambda_\pm$ to I_{mass} before performing perturbative expansion. For this reason, we define

$$\tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}] \equiv I_{mass}[g_{\mu\nu}, f_{\mu\nu}] + M_{Pl}^2 \int d^4x \sqrt{-g} \Lambda_\pm, \quad (37)$$

and expand it instead of I_{mass} itself.

As shown explicitly in Appendix A, upon using the background equation of motion for the Stückelberg fields but without using the background equation of motion for the physical metric, the graviton mass term can be expanded up to the quadratic order as

$$\begin{aligned} \tilde{I}_{mass} &= \tilde{I}_{mass}^{(0)} + \tilde{I}_{mass}^{(2)}[h_{ij}^\pi] + O(\epsilon^3), \\ \tilde{I}_{mass}^{(2)}[h_{ij}^\pi] &= \frac{M_{Pl}^2}{8} \int d^4x N a^3 \sqrt{\Omega} M_{GW}^2 [(h^\pi)^2 - h_\pi^{ij} h_{ij}^\pi], \end{aligned} \quad (38)$$

where the zero-th order part $\tilde{I}_{mass}^{(0)}$ is independent of perturbations,

$$\begin{aligned} M_{GW}^2 &\equiv \pm(r-1)m_g^2 X_\pm^2 \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}, \\ r &\equiv \frac{na}{N\alpha} = \frac{1}{X_\pm} \frac{H}{H_f}, \quad H \equiv \frac{\dot{a}}{Na}, \quad H_f \equiv \frac{\dot{\alpha}}{n\alpha}, \end{aligned} \quad (39)$$

X_\pm is given by (22), and

$$h^\pi \equiv \Omega^{ij} h_{ij}^\pi, \quad h_\pi^{ij} \equiv \Omega^{ik} \Omega^{jl} h_{kl}^\pi. \quad (40)$$

With the decomposition of h_{ij}^π in (35), the quadratic mass term is expanded as

$$\begin{aligned} \tilde{I}_{mass}^{(2)} &= M_{Pl}^2 \int d^4x N a^3 \sqrt{\Omega} M_{GW}^2 \\ &\times \left[3(\psi^\pi)^2 - \frac{1}{12} E^\pi \Delta (\Delta + 3K) E^\pi + \frac{1}{16} F_\pi^i (\Delta + 2K) F_i^\pi - \frac{1}{8} \gamma^{ij} \gamma_{ij} \right], \end{aligned} \quad (41)$$

where

$$F_\pi^i \equiv \Omega^{ij} F_j^\pi, \quad \gamma^{ik} \equiv \Omega^{jl} \Omega^{kl} \gamma_{kl}. \quad (42)$$

What is important here is that the quadratic part $\tilde{I}_{mass}^{(2)}$ is gauge-invariant and depends only on h_{ij}^π , or equivalently $(\psi^\pi, E^\pi, F_i^\pi, \gamma_{ij})$. In particular, it does not contribute to the equations of motion for ϕ and β_i .

We note that M_{GW}^2 vanishes or diverges for some special values of the parameters (α_3, α_4) :

$$\begin{aligned} \alpha_4 = -3(1 + \alpha_3), \quad \pm(\alpha_3 + 2) > 0 &\implies M_{GW}^2 = 0, \\ \alpha_4 = 1 + \alpha_3 + \alpha_3^2 &\implies M_{GW}^2 = 0, \\ \alpha_4 \rightarrow -\alpha_3, \quad \pm(1 + \alpha_3) > 0 &\implies |M_{GW}^2| \rightarrow \infty, \end{aligned} \quad (43)$$

where the \pm signs are for the \pm branches, respectively. In the following we suppose that the parameters (α_3, α_4) take generic values away from the special values shown in (43).

D. Matter perturbations and gauge-invariant variables

Let us divide matter fields σ_I ($I = 1, 2, \dots$) into the background values $\sigma_I^{(0)}$ and perturbations as

$$\sigma_I = \sigma_I^{(0)} + \delta\sigma_I. \quad (44)$$

We suppose that $\{\sigma_I\}$ forms a set of mutually independent physical degrees of freedom. Otherwise, we consider a subset of the original $\{\sigma_I\}$ consisting of independent physical degrees of freedom and rename it as $\{\sigma_I\}$. We can construct gauge-invariant variables Q_I from $\delta\sigma_I$ and metric perturbations, without referring to the Stückelberg fields.

For illustrative purpose let us decompose β_i , h_{ij} and ξ_i as

$$\begin{aligned} \beta_i &= D_i \beta + S_i, \\ h_{ij} &= 2\psi \Omega_{ij} + \left(D_i D_j - \frac{1}{3} \Omega_{ij} \Delta \right) E + \frac{1}{2} (D_i F_j + D_j F_i) + \gamma_{ij}, \\ \xi_i &= D_i \xi + \xi_i^T, \end{aligned} \quad (45)$$

where S_i , F_i and ξ_i^T are transverse, and Δ is the Laplacian associated with Ω_{ij} :

$$D^i S_i = D^i F_i = D^i \xi_i^T = 0, \quad \Delta \equiv D^i D_i. \quad (46)$$

Under the gauge transformation (31), each component of the physical metric perturbation transforms as

$$\begin{aligned}
\phi &\rightarrow \phi + \frac{1}{N}\partial_t(N\xi^0), \\
\beta &\rightarrow \beta - \frac{N}{a}\xi^0 + \frac{a}{N}\dot{\xi}, \\
\psi &\rightarrow \psi + NH\xi^0 + \frac{1}{3}\Delta\xi, \\
E &\rightarrow E + 2\xi, \\
S_i &\rightarrow S_i + \frac{a}{N}\dot{\xi}_i^T, \\
F_i &\rightarrow F_i + 2\xi_i^T, \\
\gamma_{ij} &\rightarrow \gamma_{ij}.
\end{aligned} \tag{47}$$

Noting that the vector Z^μ defined by

$$Z^0 = -\frac{a}{N}\beta + \frac{a^2}{2N^2}\dot{E}, \quad Z^i = \frac{1}{2}\Omega^{ij}(D_j E + F_j) \tag{48}$$

transforms as

$$Z^\mu \rightarrow Z^\mu + \xi^\mu, \tag{49}$$

we can construct the following gauge-invariant variables out of matter perturbations and physical metric perturbations:

$$\begin{aligned}
Q_I &\equiv \delta\sigma_I - \mathcal{L}_Z\sigma_I^{(0)}, \\
\Phi &\equiv \phi - \frac{1}{N}\partial_t(NZ^0), \\
\Psi &\equiv \psi - NHZ^0 - \frac{1}{6}\Delta E, \\
B_i &\equiv S_i - \frac{a}{2N}\dot{F}_i,
\end{aligned} \tag{50}$$

and γ_{ij} is gauge-invariant by itself. In the above, \mathcal{L}_Z is the Lie derivative along Z^μ .

Those gauge-invariant variables defined here and in Subsection III B, i.e. $\{Q_I, \Phi, \Psi, B_i, \gamma_{ij}, \phi^\pi, \beta^\pi, S_i^\pi, \psi^\pi, E^\pi, F_i^\pi\}$, are not independent. Indeed, it is easy to show that

$$\begin{aligned}
\phi^\pi &= \Phi + \frac{1}{N}\partial_t\left[\frac{1}{H}\left(\psi^\pi - \Psi - \frac{1}{6}\Delta E^\pi\right)\right], \\
\beta^\pi &= -\frac{1}{aH}\left(\psi^\pi - \Psi - \frac{1}{6}\Delta E^\pi\right) + \frac{a}{2N}\dot{E}^\pi, \\
S_i^\pi &= B_i + \frac{a}{2N}\dot{F}_i^\pi.
\end{aligned} \tag{51}$$

There are no more independent relations among gauge-invariant variables defined here and in Subsection III B.⁴ Therefore, we have the following set of independent gauge-invariant variables.

$$\{Q_I, \Phi, \Psi, B_i, \gamma_{ij}, \psi^\pi, E^\pi, F_i^\pi\}. \quad (52)$$

Based on their origins, we can divide this set of independent gauge-invariant variables into two categories as

$$\{Q_I, \Phi, \Psi, B_i, \gamma_{ij}\} \quad \text{and} \quad \{\psi^\pi, E^\pi, F_i^\pi\}. \quad (53)$$

The first category consists of those gauge-invariant variables that originate from the physical metric $g_{\mu\nu}$ and the matter fields $\{\sigma_I\}$. Thus, those in the first category already exist in GR coupled to the same matter content. On the other hand, those in the second category are physical degrees of freedom associated with the four Stückelberg fields φ^a .

E. Structure of total quadratic action

Let us now define

$$\tilde{I}[g_{\mu\nu}, \sigma_I] \equiv I_{EH, \tilde{\Lambda}}[g_{\mu\nu}] + I_{matter}[g_{\mu\nu}, \sigma_I], \quad \tilde{\Lambda} \equiv \Lambda + \Lambda_\pm, \quad (54)$$

so that

$$I = \tilde{I}[g_{\mu\nu}, \sigma_I] + \tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}]. \quad (55)$$

Since \tilde{I}_{mass} was already shown to be gauge-invariant up to the quadratic order, (55) implies that \tilde{I} is also gauge-invariant up to that order. Thus the quadratic part $\tilde{I}^{(2)}$ of \tilde{I} can be written in terms of gauge-invariant variables constructed solely from perturbations of the physical metric perturbations (ϕ, β_i, h_{ij}) and matter perturbations $\delta\sigma_I$, i.e. $\{Q_I, \Phi, \Psi, B_i, \gamma_{ij}\}$.

Therefore, the total quadratic action has the following structure.

$$I^{(2)} = \tilde{I}^{(2)}[Q_I, \Phi, \Psi, B_i, \gamma_{ij}] + \tilde{I}_{mass}^{(2)}[\psi^\pi, E^\pi, F_i^\pi, \gamma_{ij}], \quad (56)$$

where the explicit form of $\tilde{I}_{mass}^{(2)}$ is shown in (41). As already stated, gauge-invariant variables listed in (52) are independent from each other.

⁴ See also the sentence just after (44).

Note that ψ^π , E^π and F_i^π do not have kinetic terms but have nonvanishing masses, provided that the parameters (α_3, α_4) take generic values away from the special values shown in (43). Thus, we can integrate them out: their equations of motion lead to

$$\psi^\pi = E^\pi = 0, \quad F_i^\pi = 0, \quad (57)$$

and then

$$I^{(2)} = \tilde{I}^{(2)}[Q_I, \Phi, \Psi, B_i, \gamma_{ij}] - \frac{M_{Pl}^2}{8} \int d^4x N a^3 \sqrt{\Omega} M_{GW}^2 \gamma^{ij} \gamma_{ij}, \quad (58)$$

where M_{GW}^2 is given by (39). For scalar and vector modes, this quadratic action is exactly the same as that in GR with the matter content $\{\sigma_I\}$.

F. Gravitational waves with time-dependent mass

The total quadratic action for the tensor sector is

$$I_{tensor}^{(2)} = \frac{M_{Pl}^2}{8} \int d^4x N a^3 \sqrt{\Omega} \left[\frac{1}{N^2} \dot{\gamma}^{ij} \dot{\gamma}_{ij} + \frac{1}{a^2} \gamma^{ij} (\Delta - 2K) \gamma_{ij} - M_{GW}^2 \gamma^{ij} \gamma_{ij} \right], \quad (59)$$

provided that there is no tensor-type contribution from the quadratic part of I_{matter} . In this way the dispersion relation of gravitational waves is modified. The squared mass of gravitational waves M_{GW}^2 is given by (39) and is time-dependent.

If M_{GW}^2 is negative then long wavelength gravity waves exhibit linear instability. For generic values of parameters (α_3, α_4) away from the special values shown in (43), we see from the formula (39) that the sign of M_{GW}^2 is the same as the sign of the combination $\pm(r-1)m_g^2$, where \pm signs correspond to \pm branches, respectively.

IV. AN EXAMPLE: SCALAR MATTER FIELD AND MINKOWSKI FIDUCIAL

In the previous section we have analyzed quadratic action for perturbations around non-trivial FRW backgrounds, with a general FRW fiducial metric and a general matter content. For scalar and vector modes, we have shown that the quadratic action is exactly the same as that in GR with the matter content. For tensor modes, on the other hand, we have seen that gravitational waves obtain a time-dependent mass.

In this section, in order to illustrate these results a little more explicitly, we shall consider a simple example consisting of the massive gravity with Minkowski fiducial metric, coupled

to a canonical scalar matter field with potential V . The total action of this system is

$$I = \int d^4x \sqrt{-g} \left[M_{Pl}^2 \left(\frac{R}{2} + m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \right) - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\sigma) \right], \quad (60)$$

where $\mathcal{L}_{1,2,3}$ are the graviton mass terms defined in (7). In the case of the Minkowski fiducial, the only nontrivial FRW background is the open FRW solution found in [23] and reviewed in Sec. II. Thus, in this section the curvature constant K is set to be negative and the form of the fiducial metric is specified by (10)-(13).

For the FRW background (15)-(16) with $K < 0$ and $\sigma = \sigma^{(0)}(t)$, the equations of motion read

$$\begin{aligned} 3H^2 + \frac{3K}{a^2} &= \Lambda_\pm + \frac{1}{M_{Pl}^2} \left[\frac{(\dot{\sigma}^{(0)})^2}{2N^2} + V(\sigma^{(0)}) \right], \\ -\frac{2\dot{H}}{N} + \frac{2K}{a^2} &= \frac{(\dot{\sigma}^{(0)})^2}{M_{Pl}^2 N^2} \\ \frac{1}{N} \partial_t \left(\frac{\dot{\sigma}^{(0)}}{N} \right) + \frac{3H}{N} \dot{\sigma}^{(0)} + V'(\sigma^{(0)}) &= 0. \end{aligned} \quad (61)$$

We now introduce perturbations to the metric $g_{\mu\nu}$, the four Stückelberg fields φ^a and the scalar matter field σ . We will be developing the perturbation theory without specifying a gauge and in the end we will switch to gauge invariant perturbations, as we have already done in the previous section in a more general setup. The total quadratic action before switching to gauge invariant perturbations is presented in Appendix B. We then adopt the decomposition of the form

$$\begin{aligned} \pi_i &= D_i \pi + \pi_i^T, & \beta_i &= D_i \beta + S_i, \\ h_{ij} &= 2\psi \Omega_{ij} + \left(D_i D_j - \frac{1}{3} \Omega_{ij} \Delta \right) E + \frac{1}{2} (D_i F_j + D_j F_i) + \gamma_{ij}, \end{aligned} \quad (62)$$

where π_i^T , S_i and F_i are transverse:

$$D^i \pi_i^T = D^i S_i = D^i F_i = 0. \quad (63)$$

A. Tensor sector

We start by considering the tensor sector. We use the decomposition (62) in the total action (B1)-(B3), keeping only the transverse traceless mode γ_{ij} . Since the matter sector

has no tensor degrees, the action is the same as the one given in (59) with (39) and for the Minkowski fiducial, we have

$$r = \frac{aH}{\sqrt{-K}}. \quad (64)$$

Notice that in the accelerating universe, r and thus M_{GW}^2 grow at late time.

Switching to conformal time $d\eta \equiv N dt/a$ and defining the canonical fields

$$\bar{\gamma}_{ij} \equiv \frac{M_{Pl} a}{2} \gamma_{ij}, \quad (65)$$

the tensor action takes the form

$$I_{tensor}^{(2)} = \frac{1}{2} \int d^3x d\eta \sqrt{\Omega} \left[\bar{\gamma}'_{ij} \bar{\gamma}'^{ij} + \bar{\gamma}^{ij} \left(\Delta - 2K + \frac{a''}{a} \right) \bar{\gamma}_{ij} - a^2 M_{GW}^2 \bar{\gamma}^{ij} \bar{\gamma}_{ij} \right], \quad (66)$$

where a prime denotes differentiation with respect to conformal time. Next, we use harmonic expansion through

$$\bar{\gamma}_{ij} = \int k^2 dk \bar{\gamma}_{\vec{k}} Y_{ij}(\vec{k}, \vec{x}), \quad (67)$$

where $Y_{ij}(\vec{k}, \vec{x})$ is the tensor harmonic satisfying

$$\left(\Delta + \vec{k}^2 \right) Y_{ij} = 0, \quad D^i Y_{ij} = 0, \quad \Omega^{ij} Y_{ij} = 0, \quad (\vec{k}^2 \equiv \Omega_{ij} \vec{k}^i \vec{k}^j) \quad (68)$$

with $\vec{k}^2 \geq |K|$ taking continuous values. Suppressing the momentum index, we obtain the equation of motion

$$\bar{\gamma}'' + \left(\vec{k}^2 - \frac{a''}{a} + 2K + a^2 M_{GW}^2 \right) \bar{\gamma} = 0. \quad (69)$$

As we showed in the previous section for a generic setup, the tensor mode acquires a mass contribution whose time dependence is determined by the fiducial metric (which is Minkowski in the present example).

B. Vector sector

We now move on to the vector sector. Keeping only the transverse vector modes in the decomposition (62), the total action (B1)-(B3) reduces to

$$I_{vector}^{(2)} = \frac{M_{Pl}^2}{8} \int d^4x N a^3 \sqrt{\Omega} \mathcal{L}_{vector} \quad (70)$$

with

$$\begin{aligned}\mathcal{L}_{vector} = & \frac{1}{2N^2} \left(D^i \dot{F}^j D_i \dot{F}_j - 2K \dot{F}^i \dot{F}_i \right) - \frac{2}{Na} \left(D^i \dot{F}^j D_i S_j - 2K \dot{F}^i S_i \right) \\ & - \frac{1}{2} M_{GW}^2 \left(D^i F^j D_i F_j - 2K F^i F_i \right) + \frac{2}{a^2} D^i S^j D_i S_j - \frac{4K}{a^2} S_i S^i \\ & - 2M_{GW}^2 \left(\Delta \pi_i^T + 2K \pi_i^T \right) \left(F^i - \pi_T^i \right),\end{aligned}\quad (71)$$

where

$$S^i \equiv \Omega^{ij} S_j, \quad F^i \equiv \Omega^{ij} F_j, \quad \pi_T^i \equiv \Omega^{ij} \pi_j^T. \quad (72)$$

We then switch to gauge invariant perturbations, as defined in (34)-(35), and obtain

$$\begin{aligned}\mathcal{L}_{vector} = & \frac{1}{2N^2} \left(D^i \dot{F}_\pi^j D_i \dot{F}_j^\pi - 2K \dot{F}_\pi^i \dot{F}_i^\pi \right) - \frac{2}{Na} \left(D^i \dot{F}_\pi^j D_i S_j^\pi - 2K \dot{F}_\pi^i S_i^\pi \right) \\ & - \frac{1}{2} M_{GW}^2 \left(D^i F_\pi^j D_i F_j^\pi - 2K F_\pi^i F_i^\pi \right) + \frac{2}{a^2} D^i S_\pi^j D_i S_j^\pi - \frac{4K}{a^2} S_\pi^i S_i^\pi,\end{aligned}\quad (73)$$

which is manifestly gauge invariant. Varying this action with respect to S_i^π yields an algebraic equation for S_i^π , which can be solved by

$$S_i^\pi = \frac{a}{2N} \dot{F}_i^\pi. \quad (74)$$

Using this solution back in the action, we get

$$I_{vector}^{(2)} = \frac{M_{Pl}^2}{16} \int d^4x N a^3 \sqrt{\Omega} M_{GW}^2 F_\pi^i (\Delta + 2K) F_i^\pi. \quad (75)$$

This clearly shows that the kinetic term for vector perturbation vanishes at quadratic order. Provided that the parameters (α_3, α_4) take generic values away from the special values shown in (43), the equation of motion for F_i^π leads to $F_i^\pi = 0$ and then $I_{vector}^{(2)} = 0$.

C. Scalar sector

Finally, we consider the scalar perturbations. Using the decomposition (62) in the total action (B1)-(B3), we obtain the action for the scalar sector as

$$I_{scalar}^{(2)} = \frac{M_{Pl}^2}{2} \int d^4x N a^3 \sqrt{\Omega} \mathcal{L}_{scalar}, \quad (76)$$

with

$$\begin{aligned}
\mathcal{L}_{scalar} = & \frac{1}{M_{Pl}^2} \left[\frac{1}{N^2} \delta \dot{\sigma}^2 - \frac{1}{a^2} D_i \delta \sigma D^i \delta \sigma - V'' \delta \sigma^2 - \frac{2 \dot{\sigma}}{N^2} (\phi - 3 \psi) \delta \dot{\sigma} - 2 V' (\phi + 3 \psi) \delta \sigma \right] \\
& + \frac{1}{N^2} \left(\frac{1}{6} (\Delta \dot{E})^2 - \frac{K}{2} D^i \dot{E} D_i \dot{E} - 6 \dot{\psi}^2 \right) + \frac{12 H}{N} \phi \dot{\psi} - \frac{12 K}{a^2} \phi \psi \\
& - \frac{1}{6} \left(M_{GW}^2 + \frac{K}{a^2} \right) (\Delta E)^2 + \frac{K}{2} M_{GW}^2 D_i E D^i E - \left(6 H^2 - \frac{\dot{\sigma}^2}{M_{Pl}^2 N^2} \right) \phi^2 \\
& - \frac{2 K}{a^2} D_i \beta D^i \beta + \frac{2}{a N} \left(2 \Delta \dot{\psi} - \frac{1}{3} \Delta^2 \dot{E} - K \Delta \dot{E} - 2 N H \Delta \phi + \frac{\dot{\sigma}}{M_{Pl}^2} \Delta \delta \sigma \right) \beta \\
& - \frac{2}{a^2} \left(2 \Delta \psi - \frac{1}{3} \Delta^2 E - K \Delta E \right) \phi + 6 \left(M_{GW}^2 - \frac{K}{a^2} \right) \psi^2 \\
& + \frac{1}{a^2} \left(2 D_i \psi D^i \psi + \frac{2}{3} \Delta \psi \Delta E + \frac{1}{18} D_i \Delta E D^i \Delta E - 2 K D_i \psi D^i E \right) \\
& + 2 M_{GW}^2 \left(K (D_i \pi D^i \pi - D_i E D^i \pi) + \frac{1}{3} \Delta E \Delta \pi - 2 \psi \Delta \pi - 6 H N \psi \pi^0 \right) \\
& + 2 H N^2 M_{GW}^2 \left(\frac{2}{N} \Delta \pi + 3 H \pi^0 \right) \pi^0. \tag{77}
\end{aligned}$$

Hereafter in this subsection, σ represents the background value $\sigma^{(0)}(t)$. Next, we switch to gauge invariant variables defined in (34)-(35) and carry out harmonic expansion. We then obtain the equations of motion for the nondynamical degrees as

$$\begin{aligned}
\phi^\pi &= \frac{1}{H} \left(\frac{K}{a} \beta^\pi + \frac{\dot{\psi}^\pi}{N} + \frac{k^2 - 3 K}{6 N} \dot{E}^\pi + \frac{\dot{\sigma}}{2 M_{Pl}^2 N} \delta \sigma^\pi \right), \\
\beta^\pi &= -\frac{a}{k^2 H} \left[\left(-3 H^2 + \frac{\dot{\sigma}^2}{2 M_{Pl}^2 N^2} \right) \phi^\pi + \frac{k^2 - 3 K}{a^2} \psi^\pi + \frac{3 H}{N} \dot{\psi}^\pi + \frac{k^2 (k^2 - 3 K)}{6 a^2} E^\pi \right. \\
&\quad \left. - \frac{V'}{2 M_{Pl}^2} \delta \sigma^\pi - \frac{\dot{\sigma}}{2 M_{Pl}^2 N^2} \delta \dot{\sigma}^\pi \right], \tag{78}
\end{aligned}$$

where

$$\delta \sigma^\pi \equiv \delta \sigma - \dot{\sigma} \pi^0, \tag{79}$$

and substitute their solutions into the action. Defining the analogue of the Sasaki-Mukhanov variable

$$Q \equiv \delta \sigma^\pi - \frac{\dot{\sigma}}{N H} \left(\psi^\pi + \frac{k^2}{6} E^\pi \right), \tag{80}$$

the resulting quadratic action reads, up to boundary terms,

$$I_{scalar}^{(2)} = \frac{M_{Pl}^2}{2} \int d^3 k dt a^3 N \left[6 M_{GW}^2 |\psi^\pi|^2 - \frac{1}{6} M_{GW}^2 k^2 (k^2 - 3 K) |E^\pi|^2 + \mathcal{L}_Q \right], \tag{81}$$

where

$$\mathcal{L}_Q \equiv \frac{2 H^2 (k^2 - 3 K)}{2 H^2 M_{Pl}^2 (k^2 - 3 K) + K \frac{\dot{\sigma}^2}{N^2}} \left(\frac{1}{N^2} |\dot{Q}|^2 - M_Q^2 |Q|^2 \right), \quad (82)$$

and

$$M_Q^2 \equiv V'' + \frac{k^2}{a^2} \left(1 - \frac{a^2 \dot{\sigma}^2}{K M_{Pl}^2 N^2} \right) + \frac{2 (k^2 H \frac{\dot{\sigma}}{N} + K V') \left[\left(k^2 H^2 - \frac{K^2}{a^2} \right) \frac{\dot{\sigma}}{N} + K H V' \right]}{K M_{Pl}^2 H \left(2 H^2 (k^2 - 3 K) + K \frac{\dot{\sigma}^2}{M_{Pl}^2 N^2} \right)}. \quad (83)$$

Provided that the parameters (α_3, α_4) take generic values away from the special values shown in (43), the equation of motion for ψ^π and E^π lead to $\psi^\pi = E^\pi = 0$ and then

$$I_{scalar}^{(2)} = \frac{M_{Pl}^2}{2} \int d^3 k dt a^3 N \mathcal{L}_Q. \quad (84)$$

As shown in Appendix C, the action (84) agrees with the standard results in GR coupled to the same scalar matter field σ .

To summarize, the scalar sector consists of a dynamical degree which evolves exactly like the standard Sasaki-Mukhanov variable in GR, and two more degrees which have infinite mass.

V. SUMMARY AND DISCUSSIONS

In the context of the potentially ghost-free, nonlinear massive gravity [9] and for a general fiducial metric of FRW form, we analyzed linear perturbations around self-accelerating cosmological solutions of arbitrary spatial curvature, populated by generic matter content that is minimally coupled to gravity. By constructing a gauge invariant formulation of perturbations, we found that massive graviton modes in the scalar and vector sectors have vanishing kinetic terms but nonzero mass terms. By integrating them out, we showed that the part of the action quadratic in scalar and vector perturbations is exactly the same as in GR with the same matter content. In other words, the dynamical degrees in the gravity sector comprise only the two gravity wave polarizations. We also found that these acquire a mass whose time dependence is set by the fiducial metric.

In Fierz-Pauli theory in de Sitter background, it has been known that the scalar mode among five degrees of freedom of massive spin-2 graviton becomes ghost for $2H^2 > m_{FP}^2$, where H is the Hubble expansion rate and m_{FP} is the graviton mass [31]. This conclusion does not hold in the nontrivial cosmological branches of the nonlinear massive gravity. Indeed, as stated above, the scalar and vector modes have vanishing kinetic terms and nonzero

mass terms for any FRW background. This sharp contrast to the linear (Fierz-Pauli) massive gravity stems from a peculiar structure of the graviton mass term expanded up to the quadratic order in perturbations: it depends only on the (ij) -components of metric perturbations and thus are independent of (00) and $(0i)$ -components. This Lorentz-violating structure is possible because the vev of $f_{\mu\nu}$ in the cosmological branches spontaneously breaks diffeomorphism invariance in a nontrivial way.

It is still fair to say that the nature of the cancellation of the kinetic terms in the quadratic action of scalar and vector sectors is not well understood. This may be an indication that these sectors exhibit an infinitely strong coupling. If this is the case then we cannot properly describe the scalar and vector sectors without knowledge of a UV completion. On the other hand, since the modes have non vanishing masses, it may be possible that these are just infinitely heavy modes without low-energy dynamics, as we have assumed in the main part of the present paper. In this case we can safely integrate out those extra modes and trust the resulting low energy effective theory. In order to establish the fate of those extra degrees of freedom, i.e to judge whether they are strongly-coupled or non-dynamical, the linear perturbation theory is not sufficient and nonlinear methods are needed. This study is beyond the scope of the present paper and is left for a future work.

On the other hand, the modification in the tensor sector may leave a signature in the stochastic gravitational wave spectrum. The additional term in the mass of the tensor modes is time dependent, while its sign is determined by both the fiducial metric and the cosmological evolution. A positive but large contribution may give rise to a suppression of the gravity waves and a null signal in the large scale tensor-to-scalar ratio. However, this deviation from scale invariance may allow the signal at small scales to be potentially observable in the space-based gravity wave observatories such as DECIGO [32], BBO [33] and LISA [34].

It is important to note that the present analysis is purely classical and special care is needed when discussing the evolution of cosmological perturbations which start off in quantum mechanical vacuum. In order to address quantum issues such as the radiative stability of the structure of the effective theory describing cosmological perturbations, one of the first important steps is to identify the strong coupling scale below which the nonlinear massive gravity theory can be trusted. In the branch described by the trivial solution to the Stückelberg equation of motion (20), the so called decoupling limit has been useful for

this purpose. However, since this trivial branch is not compatible with FRW cosmologies, in the present paper we have considered other two branches described by non-trivial solutions (22). In these cosmological branches, the usual decoupling limit is not applicable, at least apparently. We thus need to develop a new technique to identify the strong coupling scale, or directly analyze nonlinear dynamics of the whole system. This is certainly one of the most important issues in the future research.

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Appendix A: Calculation of graviton mass term

In this appendix we consider the graviton mass term introduced in Sec. II and generalized at the beginning of Sec. III. We thus consider arbitrary K , $n(\varphi^0)$ and $\alpha(\varphi^0)$. Without loss of generality, we assume that $N > 0$, $n > 0$, $a > 0$ and $\alpha > 0$ at least in the vicinity of the time of interest, where N and a are the lapse function and the scale factor for the background FRW physical metric.

We expand the generalized graviton mass term up to the quadratic order in perturbations. In doing so, we shall not use the background equations of motion for the physical metric since they depend not only on the graviton mass term but also on the Einstein-Hilbert action and the matter action. On the other hand, we shall use the background equation of motion for the Stückelberg fields in the middle of calculation.

1. Background equation of motion for Stückelberg fields

In this subsection we shall derive the background equation of motion for the Stückelberg fields φ^a by expanding the graviton mass term I_{mass} up to the linear order in π^a ($= \delta\varphi^a$)

without variation of the physical metric $g_{\mu\nu}$. This will be a good warm-up for the forthcoming subsections.

Using the formula (29), $f_{\mu\nu}$ is expanded up to the linear order as

$$\begin{aligned} f_{00} &= -n^2 \left[1 + \frac{2}{n} \partial_t(n\pi^0) + O(\epsilon^2) \right], \\ f_{0i} &= f_{i0} = \alpha n \left[-\frac{n}{\alpha} D_i \pi^0 + \frac{\alpha}{n} \dot{\pi}_i + O(\epsilon^2) \right], \\ f_{ij} &= \alpha^2 \left[(1 + 2nH_f \pi^0) \Omega_{ij} + D_i \pi_j + D_j \pi_i + O(\epsilon^2) \right], \end{aligned} \quad (\text{A1})$$

where an overdot represents differentiation with respect to the time t and

$$H_f \equiv \frac{\dot{\alpha}}{n\alpha}. \quad (\text{A2})$$

With the unperturbed physical metric

$$g_{00} = -N^2, \quad g_{0i} = g_{i0} = 0, \quad g_{ij} = a^2 \Omega_{ij}, \quad (\text{A3})$$

this leads to the following expansion for f^μ_ν ($\equiv g^{\mu\rho} f_{\rho\nu}$).

$$\begin{aligned} f^0_0 &= \frac{n^2}{N^2} \left[1 + \frac{2}{n} \partial_t(n\pi^0) + O(\epsilon^2) \right], \\ f^0_i &= -\frac{\alpha n}{N^2} \left[-\frac{n}{\alpha} D_i \pi^0 + \frac{\alpha}{n} \dot{\pi}_i + O(\epsilon^2) \right], \\ f^i_0 &= \frac{\alpha n}{a^2} \left[-\frac{n}{\alpha} D^i \pi^0 + \frac{\alpha}{n} \dot{\pi}^i + O(\epsilon^2) \right], \\ f^i_j &= \frac{\alpha^2}{a^2} \left[(1 + 2nH_f \pi^0) \delta^i_j + D^i \pi_j + D_j \pi^i + O(\epsilon^2) \right]. \end{aligned} \quad (\text{A4})$$

Then, using the formula in Appendix D for matrix square-root, \mathcal{K}^μ_ν defined by (8)-(9) is expanded up to the linear order as

$$\mathcal{K}^\mu_\nu = \mathcal{K}^{(0)\mu}_\nu + \mathcal{K}^{(1)\mu}_\nu + O(\epsilon^2), \quad (\text{A5})$$

where

$$\mathcal{K}^{(0)0}_0 = 1 - \frac{n}{N}, \quad \mathcal{K}^{(0)0}_i = 0, \quad \mathcal{K}^{(0)i}_0 = 0, \quad \mathcal{K}^{(0)i}_j = \left(1 - \frac{\alpha}{a}\right) \delta^i_j, \quad (\text{A6})$$

and

$$\begin{aligned} \mathcal{K}^{(1)0}_0 &= -\frac{1}{N} \partial_t(n\pi^0), \\ \mathcal{K}^{(1)0}_i &= \frac{na}{N^2(1+r)} \left[-\frac{n}{\alpha} D_i \pi^0 + \frac{\alpha}{n} \dot{\pi}_i \right], \\ \mathcal{K}^{(1)i}_0 &= -\frac{n}{a(1+r)} \left[-\frac{n}{\alpha} D^i \pi^0 + \frac{\alpha}{n} \dot{\pi}^i \right], \\ \mathcal{K}^{(1)i}_j &= -\frac{\alpha}{2a} \left[2nH_f \pi^0 \delta^i_j + D^i \pi_j + D_j \pi^i \right]. \end{aligned} \quad (\text{A7})$$

Here, we have defined

$$r \equiv \frac{na}{N\alpha}. \quad (\text{A8})$$

It is now straightforward to expand the graviton mass term (6) up to the first order. The result is

$$I_{mass} = I_{mass}^{(0)} + M_{Pl}^2 m_g^2 \int d^4x N a^3 \sqrt{\Omega} \frac{3n}{a} (aH - \alpha H_f) J_\phi \pi^0 + O(\epsilon^2), \quad (\text{A9})$$

where the zero-th order part $I_{mass}^{(0)}$ does not depend on π^a and

$$J_\phi \equiv 3 - 2X + \alpha_3(1 - X)(3 - X) + \alpha_4(1 - X)^2, \quad X \equiv \frac{\alpha}{a}. \quad (\text{A10})$$

Therefore, the background equation of motion for the Stückelberg fields is

$$(aH - \alpha H_f) J_\phi = 0, \quad (\text{A11})$$

where H_f is defined in (A2) and

$$H \equiv \frac{\dot{a}}{Na}. \quad (\text{A12})$$

Setting $aH = \alpha H_f$ would not allow nontrivial cosmologies since in this case the background evolution of the physical metric would be determined not by the matter content but by the fiducial metric. Thus, we restrict our attention to solutions of $J_\phi = 0$. This leads to $X = X_\pm$, where X_\pm are given by (22).

2. Unitary gauge

In this subsection we shall expand the mass term up to the quadratic order in the unitary gauge, i.e. under the gauge condition

$$\pi^0 = \pi^i = 0. \quad (\text{A13})$$

After obtaining the expression in the unitary gauge, it is relatively easy to infer the corresponding expression in general gauge and to confirm it. The expression in general gauge will be presented in the next subsection.

In the unitary gauge, the fiducial metric $f_{\mu\nu}$ is the same as in the background. Hence,

$f^\mu_\nu (\equiv g^{\mu\rho} f_{\rho\nu})$ is expanded up to the second order as

$$\begin{aligned} f^0_0 &= \frac{n^2}{N^2} [1 - 2\phi + (4\phi^2 - \beta^i\beta_i) + O(\epsilon^3)], \\ f^0_i &= \frac{\alpha^2}{Na} [\beta_i - 2\phi\beta_i - \beta^j h_{ji} + O(\epsilon^3)], \\ f^i_0 &= -\frac{n^2}{Na} [\beta^i - 2\phi\beta^i - \beta_j h^{ji} + O(\epsilon^3)], \\ f^i_j &= \frac{\alpha^2}{a^2} [\delta^i_j - h^i_j - \beta^i\beta_j + h^{ik}h_{kj} + O(\epsilon^3)], \end{aligned} \quad (\text{A14})$$

where

$$\beta^i \equiv \Omega^{ij}\beta_j, \quad h^i_j = h_j^i \equiv \Omega^{ik}h_{kj}, \quad h^{ij} \equiv \Omega^{ik}\Omega^{jl}h_{kl}. \quad (\text{A15})$$

Then, taking the matrix square-root as prescribed in Appendix D, \mathcal{K}^μ_ν defined by (8)-(9) is expanded up to the quadratic order as

$$\mathcal{K}^\mu_\nu = \mathcal{K}^{(0)\mu}_\nu + \mathcal{K}^{(1)\mu}_\nu + \mathcal{K}^{(2)\mu}_\nu + O(\epsilon^3), \quad (\text{A16})$$

where

$$\mathcal{K}^{(0)0}_0 = 1 - \frac{n}{N}, \quad \mathcal{K}^{(0)0}_i = 0, \quad \mathcal{K}^{(0)i}_0 = 0, \quad \mathcal{K}^{(0)i}_j = \left(1 - \frac{\alpha}{a}\right) \delta^i_j, \quad (\text{A17})$$

$$\mathcal{K}^{(1)0}_0 = \frac{n}{N}\phi, \quad \mathcal{K}^{(1)0}_i = -\frac{\alpha\beta_i}{N(1+r)}, \quad \mathcal{K}^{(1)i}_0 = \frac{nr\beta^i}{a(1+r)}, \quad \mathcal{K}^{(1)i}_j = \frac{\alpha}{2a}h^i_j, \quad (\text{A18})$$

and

$$\begin{aligned} \mathcal{K}^{(2)0}_0 &= \frac{n}{N} \left[-\frac{3}{2}\phi^2 + \frac{r(2+r)}{2(1+r)^2} \beta^k\beta_k \right], \\ \mathcal{K}^{(2)0}_i &= \frac{\alpha}{N(1+r)} \left[\frac{2+r}{1+r} \phi\beta_i + \frac{1+2r}{2(1+r)} \beta^k h_{ki} \right], \\ \mathcal{K}^{(2)i}_0 &= -\frac{nr}{a(1+r)} \left[\frac{2+r}{1+r} \phi\beta^i + \frac{1+2r}{2(1+r)} \beta_k h^{ki} \right], \\ \mathcal{K}^{(2)i}_j &= \frac{\alpha}{a} \left[\frac{1+2r}{2(1+r)^2} \beta^i\beta_j - \frac{3}{8} h^{ik} h_{kj} \right]. \end{aligned} \quad (\text{A19})$$

It is then straightforward to expand $[\mathcal{K}^n]$ ($n = 1, 2, 3, 4$) up to the quadratic order. The result is

$$[\mathcal{K}^n] = [\mathcal{K}^n]^{(0)} + [\mathcal{K}^n]^{(1)} + [\mathcal{K}^n]^{(2)} + O(\epsilon^3), \quad (\text{A20})$$

where

$$\begin{aligned} [\mathcal{K}^n]^{(0)} &= 3(1-X)^n + (1-rX)^n, \\ [\mathcal{K}^n]^{(1)} &= nrX(1-rX)^{n-1}\phi + \frac{n}{2}X(1-X)^{n-1}h, \end{aligned} \quad (\text{A21})$$

and

$$\begin{aligned}
[\mathcal{K}]^{(2)} &= -\frac{3}{2}rX\phi^2 - \frac{3}{8}Xh^{ij}h_{ij} + \frac{r_2X}{2r_1}\beta^i\beta_i, \\
[\mathcal{K}^2]^{(2)} &= (4rX - 3)rX\phi^2 + \left(X - \frac{3}{4}\right)Xh^{ij}h_{ij} + \frac{X}{r_1}(r_2 - r_3X)\beta^i\beta_i, \\
[\mathcal{K}^3]^{(2)} &= -\frac{3}{2}(3 - 5rX)(1 - rX)rX\phi^2 - \frac{3}{8}(3 - 5X)(1 - X)Xh^{ij}h_{ij} \\
&\quad + \frac{3X}{2r_1}(r_2 - 2r_3X + r_4X^2)\beta^i\beta_i, \\
[\mathcal{K}^4]^{(2)} &= 6(2rX - 1)(1 - rX)^2rX\phi^2 + \frac{3}{2}(2X - 1)(1 - X)^2Xh^{ij}h_{ij} \\
&\quad + \frac{2X}{r_1}(r_2 - 3r_3X + 3r_4X^2 - r_5X^3)\beta^i\beta_i.
\end{aligned} \tag{A22}$$

Here, X is defined in (A10) and

$$r_n \equiv \sum_{i=0}^n r^n. \tag{A23}$$

We are now ready to expand the graviton mass term (6) up to the quadratic order. However, before doing so, let us expand it up to the linear order in order to see the effective energy density due to the graviton mass term. The result is

$$I_{mass} = I_{mass}^{(0)} + \int dx^4 Na^3 \sqrt{\Omega} \left[-\left(\phi + \frac{1}{2}h\right) \rho_g + \frac{1}{2}M_{Pl}^2 m_g^2 (1 - r)XhJ_\phi \right] + O(\epsilon^2), \tag{A24}$$

where the zero-th order part $I_{mass}^{(0)}$ does not depend on perturbations, and

$$\rho_g = -M_{Pl}^2 m_g^2 (1 - X) [3(2 - X) + \alpha_3(1 - X)(4 - X) + \alpha_4(1 - X)^2] \tag{A25}$$

is the effective energy density due to the graviton mass.

Having obtained the expression for ρ_g and noticing that the factor $(\phi + h/2)$ in (A24) is the linear order part of

$$\frac{\sqrt{-g}}{Na^3 \sqrt{\Omega}} = 1 + \left(\phi + \frac{1}{2}h\right) + \left[-\frac{1}{2}\phi^2 + \frac{1}{2}\beta^i\beta_i + \frac{1}{8}(h^2 - 2h^{ij}h_{ij}) + \frac{1}{2}\phi h\right] + O(\epsilon^3), \tag{A26}$$

we expect that expanding

$$\tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}] \equiv I_{mass}[g_{\mu\nu}, f_{\mu\nu}] + \int d^4x \sqrt{-g} \rho_g, \tag{A27}$$

instead of I_{mass} itself, should simplify the resulting expression. We thus expand \tilde{I}_{mass} up to the quadratic order. The result is

$$\tilde{I}_{mass} = M_{Pl}^2 m_g^2 \int d^4x Na^3 \sqrt{\Omega} \left[\tilde{L}_{mass}^{(0)} + \tilde{L}_{mass}^{(1)} + \tilde{L}_{mass}^{(2)} + O(\epsilon^3) \right], \tag{A28}$$

where

$$\begin{aligned}
\tilde{L}_{mass}^{(0)} &= -rX(1-X) [3 + 3\alpha_3(1-X) + \alpha_4(1-X)^2], \\
\tilde{L}_{mass}^{(1)} &= \frac{1}{2}(1-r)XhJ_\phi, \\
\tilde{L}_{mass}^{(2)} &= \frac{1}{2} \left[\phi h + \frac{\beta^i \beta_i}{1+r} + \frac{1}{4}(1-r)(h^2 - 2h^{ij}h_{ij}) \right] XJ_\phi \\
&\quad + \frac{1}{8}m_g^{-2}M_{GW}^2(h^2 - h^{ij}h_{ij}),
\end{aligned} \tag{A29}$$

and

$$m_g^{-2}M_{GW}^2 = XJ_\phi + (1-r)X^2[1 + \alpha_3(2-X) + \alpha_4(1-X)]. \tag{A30}$$

As shown in the previous subsection, the background equation of motion for the Stückelberg fields for nontrivial cosmological branches is $J_\phi = 0$ and gives $X = X_\pm$, where J_ϕ and X_\pm are defined in (A10) and (22), respectively. For $X = X_\pm$, it is easy to see that $\rho_g = M_{Pl}^2\Lambda_\pm$, where Λ_\pm are defined in (24). Thus, upon using $J_\phi = 0$, the definition of \tilde{I}_{mass} in (A27) reduces to (37). Also, for $X = X_\pm$, M_{GW}^2 defined above reduces to that defined in (39).

In summary, in the unitary gauge, upon using the background equation of motion $J_\phi = 0$ for the Stückelberg fields but without using the background equation of motion for the physical metric, \tilde{I}_{mass} defined in (37) is expanded up to the quadratic order as

$$\tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}] = \tilde{I}_{mass}^{(0)} + \frac{M_{Pl}^2}{8} \int d^4x Na^3 \sqrt{\Omega} M_{GW}^2 (h^2 - h^{ij}h_{ij}) + O(\epsilon^3), \tag{A31}$$

where the zero-th order part $\tilde{I}_{mass}^{(0)} \equiv M_{Pl}^2 m_g^2 \int d^4x Na^3 \sqrt{\Omega} \tilde{L}_{mass}^{(0)}$ does not depend on the perturbations.

3. General gauge

In general gauge, we expect that the expansion of \tilde{I}_{mass} defined in (A27) should be similar to that in the unitary gauge, provided that each metric perturbation variable is replaced by the corresponding gauge-invariant variable constructed from the metric perturbation and the Stückelberg field perturbation. Such gauge-invariant variables are defined in (34). We thus expect that

$$\tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}] = \tilde{I}_{mass}^{(0)} + \frac{M_{Pl}^2}{8} \int d^4x Na^3 \sqrt{\Omega} M_{GW}^2 [(h^\pi)^2 - h_\pi^{ij}h_{ij}^\pi] + \Delta_{mass}, \tag{A32}$$

with relatively simple expression for Δ_{mass} up to the quadratic order. Of course, for $J_\phi = \pi^a = 0$, Δ_{mass} should vanish up to the quadratic order, as shown in the previous subsection. Indeed, by direct computation we can confirm that this is true not only in unitary gauge $\pi^a = 0$, but also for $\pi^a \neq 0$ as far as $J_\phi = 0$ is imposed:

$$\Delta_{mass} = O(\epsilon^3), \quad \text{for } J_\phi = 0. \quad (\text{A33})$$

If we do not impose $J_\phi = 0$ nor $\pi^a = 0$ then Δ_{mass} up to the quadratic order is

$$\begin{aligned} \Delta_{mass} = & M_{Pl}^2 m_g^2 \int dx^4 N a^3 \sqrt{\Omega} \left\{ \left[\frac{1-r}{2} X h^\pi - 3 \dot{X} \pi^0 \right] \right. \\ & + \frac{1}{2} \left[\phi^\pi h^\pi + \frac{\beta_\pi^i \beta_i^\pi}{1+r} + \frac{1}{4} (1-r) (h^2 - 2 h^{ij} h_{ij}) \right. \\ & - (1-r) (\dot{h} + 3 N H h) \pi^0 - \frac{2N}{a} (1-r) \beta^i D_i \pi^0 \\ & - \frac{a^2}{N^2 (1+r)} D^i \dot{\pi}^0 D_i \dot{\pi}^0 - \frac{N^2}{a^2} (1-r) D^i \pi^0 D_i \pi^0 + \frac{a^2}{N^2 (1+r)} \dot{\pi}^i \dot{\pi}_i \\ & \left. \left. + 3(1-r) \pi^0 \left(\frac{1}{a^3} \partial_t (N a^3 H \pi^0) + N H D_i \pi^i \right) \right] X \right. \\ & \left. + \frac{3}{2} \left[-(2\phi + h) + \frac{1}{N a^3} \partial_t (N a^3 \pi^0) + D_i \pi^i \right] \pi^0 \dot{X} \right\} J_\phi + O(\epsilon^3). \quad (\text{A34}) \end{aligned}$$

Appendix B: Quadratic action for the scalar field example

This appendix presents the total quadratic action for the scalar field example of Sec. IV. Combining the graviton mass term (A32)-(A33) with the Einstein-Hilbert and scalar field parts, the complete quadratic action can be calculated up to boundary terms as

$$\begin{aligned} I^{(2)} = & M_{Pl}^2 \int d^4 x N a^3 \sqrt{\Omega} \left\{ \mathcal{L} + \frac{\sqrt{-g}^{(2)}}{N a^3 \sqrt{\Omega}} \left[3 H^2 + \frac{3 K}{a^2} - \Lambda_\pm - \frac{1}{M_{Pl}^2} \left(\frac{\dot{\sigma}^2}{2 N^2} + V \right) \right] \right. \\ & \left. + \frac{1}{8} (h^2 - 2 h_{ij} h^{ij}) \left(\frac{2 \dot{H}}{N} - \frac{2 K}{a^2} + \frac{\dot{\sigma}^2}{M_{Pl}^2 N^2} \right) \right\}, \quad (\text{B1}) \end{aligned}$$

where

$$\frac{\sqrt{-g}^{(2)}}{N a^3 \sqrt{\Omega}} = -\frac{\phi^2}{2} + \frac{1}{2} \beta_i \beta^i - \frac{1}{4} h_{ij} h^{ij} + \frac{1}{8} h^2 + \frac{1}{2} \phi h, \quad (\text{B2})$$

and

$$\begin{aligned}
\mathcal{L} = & \frac{1}{8N^2} \left(h_{ij} \dot{h}^{ij} - \dot{h}^2 \right) + \frac{H}{N} \phi \dot{h} - \frac{1}{a} \left(2H\phi - \frac{1}{2N} \dot{h} \right) D_i \beta^i - \frac{1}{2Na} D_i \beta_j \dot{h}^{ij} - 3H^2 \phi^2 \\
& + \frac{1}{4a^2} [D_i \beta_j D^i \beta^j - (D_i \beta^i)^2 - 2K \beta^i \beta_i] + \frac{1}{2a^2} (D_i D_j h^{ij} - \Delta h) \phi \\
& + \frac{1}{8a^2} [2D^i h_{ik} D_j h^{jk} - D_k h_{ij} D^k h^{ij} + 2h D_i D_j h^{ij} - h \Delta h] - \frac{K}{4a^2} (h_{ij} h^{ij} + 4h\phi) \\
& + \mathcal{M}_{GW}^2 \left[\frac{1}{8} (h^2 - h_{ij} h^{ij}) - \frac{1}{2} (D_i \pi^i + 2NH\pi^0) h + \frac{1}{2} h_{ij} D^i \pi^j + 3H^2 N^2 (\pi^0)^2 \right. \\
& \quad \left. + \frac{1}{4} [(D_i \pi^i)^2 - D_i \pi_j D^i \pi^j + 2K \pi_i \pi^i] + 2HN\pi^0 D_i \pi^i \right] \\
& + \frac{1}{M_{Pl}^2} \left[\frac{\delta \dot{\sigma}^2}{2N^2} + \frac{\dot{\sigma}}{2N^2} (h - 2\phi) \delta \dot{\sigma} - \frac{1}{2a^2} D_i \delta \sigma D^i \delta \sigma - \frac{\dot{\sigma}}{aN} \beta_i D^i \delta \sigma - \frac{V''}{2} \delta \sigma^2 \right. \\
& \quad \left. - \frac{V'}{2} (h + 2\phi) \delta \sigma + \frac{\dot{\sigma}^2}{2N^2} \phi^2 \right]. \quad (B3)
\end{aligned}$$

Appendix C: Comparison with GR

In this appendix we compare the scalar action (84) with the GR results in the literature. For this purpose it is useful to define the gauge invariant Bardeen potential, which in our language corresponds to

$$\Phi = \phi^\pi + \frac{1}{N} \frac{d}{dt} \left(a \beta^\pi - \frac{a^2}{2N} \dot{E}^\pi \right), \quad (C1)$$

or, using Eqs.(61), (78) and (80), as well as the equation of motion for Q obtained from varying (82), we can write

$$\Phi = -\frac{a^2 H^2}{2M_{Pl}^2 H^2 (k^2 - 3K) + K \frac{\dot{\sigma}^2}{N^2}} \frac{\dot{\sigma}}{N} \left[\frac{\dot{Q}}{N} + \left(3H - \frac{\dot{\sigma}^2}{2M_{Pl}^2 H N^2} + \frac{NV'}{\dot{\sigma}} \right) Q \right]. \quad (C2)$$

Using the above definition, along with the equation of motion for Q , then switching to conformal time $a d\eta = N dt$, we see that the equation of motion for Φ in the standard scenario, given in Eq.(8.140) of Ref. [35], is satisfied. To compare the normalization of the action with the literature, it is also useful to define the following quantities

$$\xi \equiv \frac{2aM_{Pl}^2}{H} \Phi, \quad \zeta \equiv \frac{NH}{\dot{\sigma}} Q - \frac{KH N^2}{a^2 \dot{\sigma}^2} \xi. \quad (C3)$$

With these definitions, it is straightforward to verify that the equations of motion given in Eqs.(21-22) of Ref. [36] are satisfied, with substitution $dt \rightarrow N dt$. Furthermore, using the above expressions, the action given in Eq. (24) of [36] reproduces our action (84) up to boundary terms.

Appendix D: Perturbative expansion of matrix square root

Let us expand an $N \times N$ matrix A as

$$A = A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + O(\epsilon^3), \quad (\text{D1})$$

where the zero-th order part is assumed to be of the form

$$(A^{(0)})^t_t = \alpha, \quad (A^{(0)})^t_j = 0, \quad (A^{(0)})^i_t = 0, \quad (A^{(0)})^i_j = \beta \delta_j^i, \quad (\text{D2})$$

with $\alpha > 0$ and $\beta > 0$. The square-root is expanded as

$$\sqrt{A} = B^{(0)} + \epsilon B^{(1)} + \epsilon^2 B^{(2)} + O(\epsilon^3), \quad (\text{D3})$$

where

$$(B^{(0)})^t_t = \sqrt{\alpha}, \quad (B^{(0)})^t_j = 0, \quad (B^{(0)})^i_t = 0, \quad (B^{(0)})^i_j = \sqrt{\beta} \delta_j^i, \quad (\text{D4})$$

$$\begin{aligned} (B^{(1)})^t_t &= \frac{1}{2\sqrt{\alpha}} (A^{(1)})^t_t, & (B^{(1)})^t_j &= \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} (A^{(1)})^t_j, \\ (B^{(1)})^i_t &= \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} (A^{(1)})^i_t, & (B^{(1)})^i_j &= \frac{1}{2\sqrt{\beta}} (A^{(1)})^i_j, \end{aligned} \quad (\text{D5})$$

and

$$\begin{aligned} (B^{(2)})^t_t &= \frac{1}{2\sqrt{\alpha}} (\tilde{A}^{(2)})^t_t, & (B^{(2)})^t_j &= \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} (\tilde{A}^{(2)})^t_j, \\ (B^{(2)})^i_t &= \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} (\tilde{A}^{(2)})^i_t, & (B^{(2)})^i_j &= \frac{1}{2\sqrt{\beta}} (\tilde{A}^{(2)})^i_j. \end{aligned} \quad (\text{D6})$$

Here, $\tilde{A}^{(2)} \equiv A^{(2)} - B^{(1)2}$.

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